Flow polytopes and triangulations of the Caracol polytope

UNIVERSIDAD SERGIO ARBOLEDA

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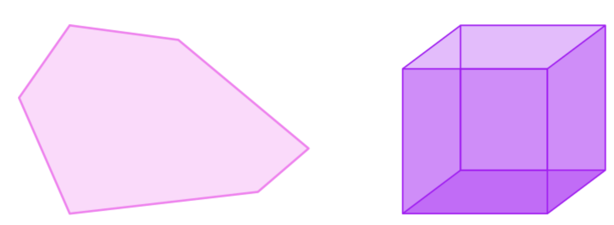
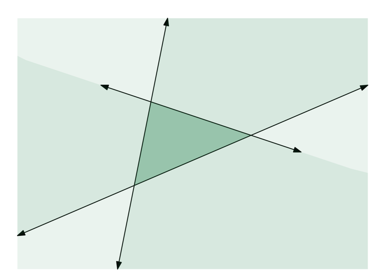
Polytopes from flow graphs

- **What is a polytope?**
  A *polytope* is basically a generalization of the concept of a polygon.



Examples of polytopes.

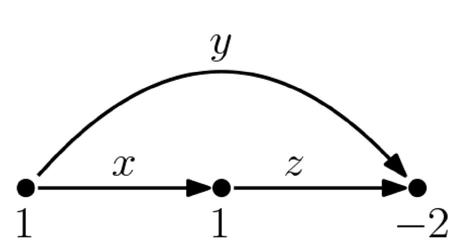


A triangle seen as an $\mathcal{H}$-polytope.

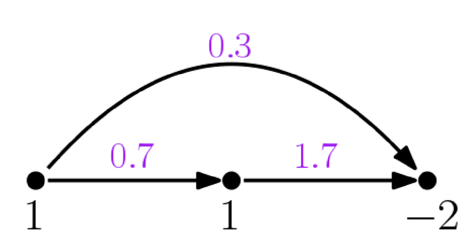
In general, we can define a polytope in two different but equivalent ways, according to their faces or according to their vertices. In the former case, we call them *$\mathcal{H}$-polytopes* and in the later we call them *$\mathcal{V}$-polytopes*.

- **How to obtain a polytope from a graph?**
  We consider a *directed graph* together with a *net flow* on its vertices. A *flow* on this graph is an assignment of nonnegative values on the edges such that every vertex is balanced. It turns out that the set of flows on a graph form a polytope determined by a set of equalities and inequalities coming from the information stored on the graph.

Example:



Graph with net flow $(1, 1, -2)$.

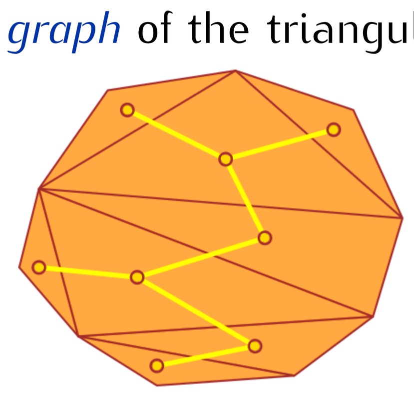


A flow.

In this example, the inequalities obtained are $x \geq 0$, $y \geq 0$ and $z \geq 0$. The equations are given by $1 = x + y$ and $x + 1 = z$.

The dual graph of a polytope triangulation

A *triangulation* of a polytope is a subdivision in which every part is a *simplex* (a polytope that generalizes a triangle) whose vertices are vertices of the original polytope.
Stanley-Postnikov and Mezsáros-Morales proposed a procedure to construct different subdivisions of a flow polytope. For net flow vector $(1, 0, \dots, 0, -1)$ any such subdivision happens to be a triangulation. This triangulation give us a new graph called the *dual graph* of the triangulation.

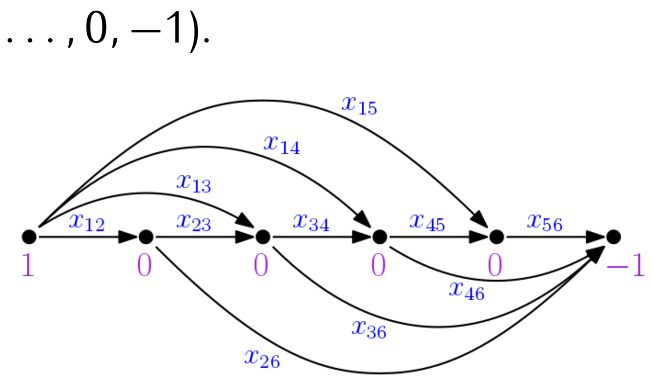


Dual graph of some triangulation.

The vertices of the dual graph correspond to the parts in the triangulation an there is an edge whenever two parts intersect maximally, that is, if two simplices in the triangulation share a common face of maximal dimension.

The caracol graph

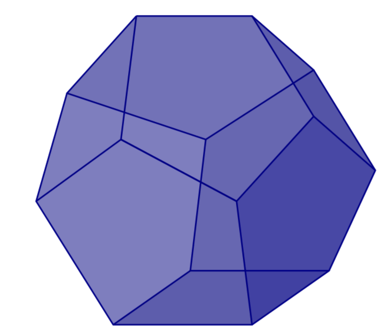
Benedetti et al. studied a particular family of flow graphs that they called, for every integer n, the *n-Caracol graph* (because of its similarity in shape with a snail). This graph has $n+1$ vertices and directed edges of the form $(1, i)$, $(i, i+1)$ and $(i, n+1)$ for each $i = 2, \dots, n$. We will consider this directed graph together with the net flow $(1, 0, \dots, 0, -1)$.



The 5-Caracol graph.

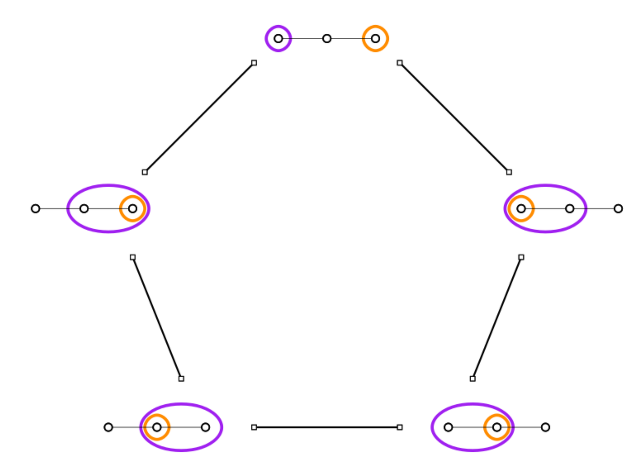
The associahedron and paths on the graph

The *associahedron* is a very popular convex polytope studied in combinatorics and algebra. This polytope has intresting properties and various, different but equivalent, representations. The number of vertices of this polytope is given by the *Catalan numbers*.

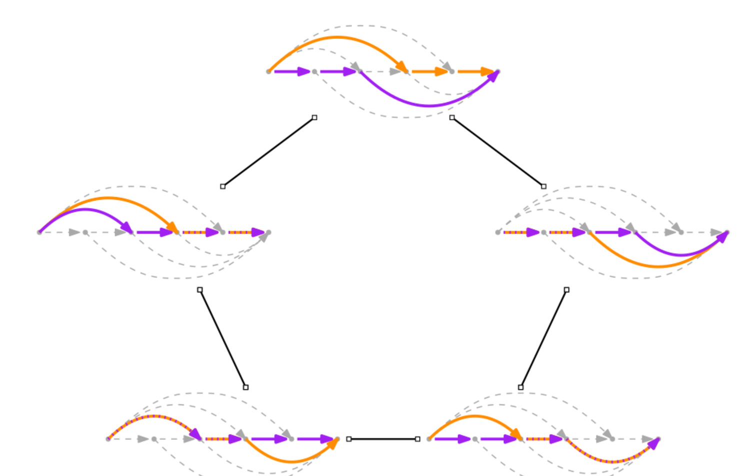


The 3-dimensional associahedron.

- **Tubings and the nested-set complex.**
  One of the multiple ways of constructing the associahedron is given by what is called a *tubing*. A tubing is a collection of proper subsets of the set of vertices forming a connected induced subgraph, called *tubes*, and such that the tubes are either nested (contained in one another) or that the union of any pair of tubes is not a tube.



Tubings.

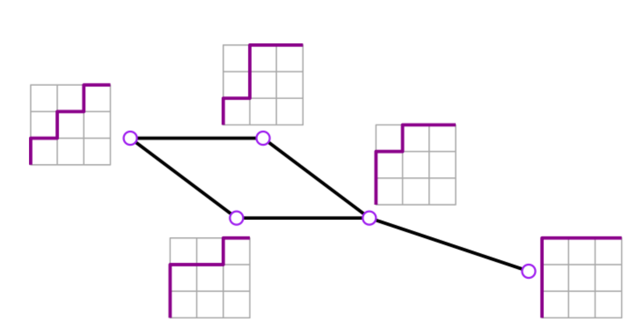


Paths.

We found a *bijection between tubings* on a line graph *and collections of compatible paths* on the Caracol graph. This tool allows us to construct the associahedron using collections of paths that at the same time provide a triangulation that can be described using the Stanley-Postnikov and Mezsáros-Morales method.

Our results

The Stanley-Postnikov and Mezsáros-Morales method depends on a particular ordering on the sets of incoming and outgoing edges at every vertex. Different orderings give different triangulations. We have the following two theorems.
Theorem 1. The dual graph of the triangulation of the Caracol flow polytope given by the reverse lexicographical ordering on the edges is the 1-Skeleton of the associahedron.
Theorem 2. The dual graph of the triangulation of the Caracol flow polytope given by the planar ordering on the edges is the toggle graph on the set of Dyck paths.



Toggle graph on the set of Dyck paths.

References

[1] Benedetti, C.; González D'León, R. S.; Hanusa, C. R. H.; Harris, P. E.; Khare, A.; Morales, A. H. and Yip, M.. *A combinatorial model for computing volumes of flow polytopes.* Trans. Amer. Math. Soc., May 23, 2019. DOI: https://doi.org/10.1090/tran/7743.

[2] Mészáros, K.; Morales, A. H. and Striker, J.. *On flow polytopes, order polytopes and certain faces of the alternating sing matrix polytope*, https://arxiv.org/abs/1510.03357v1, 2015.